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# **Symmetry group theorem to the Lin-Tsien equation and conservation laws relating to the symmetry of the equation**

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## **Abstract**

We derive the symmetry group theorem to the Lin-Tsien equation by using the modified CK's direct method, from which we obtain the corresponding symmetry group. Conservation laws corresponding to the Kac-Moody-Virasoro symmetry algebra of Lin-Tsien equation is obtained up to second order group invariants.

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## I. INTRODUCTION

Conservation law which originates in mechanics and physics play an important role in physics and mathematics. Nonlinear partial differential equations (NPDEs) that admit conservation laws arise in many disciplines of the applied sciences including physical chemistry, fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magnetohydro-dynamics, nonlinear optics, and the bio-sciences. Especially, in soliton theory, conservation laws have many significant uses, particularly with regard to integrability and linearization, analysis of solutions, and numerical solution methods. Furthermore, completely integrable NPDEs [1, 2] admit infinitely many independent conservation laws. Besides, finding the symmetry of NPDEs is also very important. The mathematical foundations for the determination of the full group for a system of differential equations can be found in Ames [3], Bluman and Cole [4], and the general theory is found in Ovsiannikov [5]. Among them, the modified CK's direct method [6, 7] is an effective method for finding symmetries and one advantage of this method is that one can easily obtain the relationship between new exact solutions and old ones of the given NPDEs.

Conservation law is closely connected with the the symmetry and this connection is given by the famous Noether theorem. In the classical Noethers theorem [8], if a given system of differential equations has a variational principle, then a continuous symmetry (point, contactor higher order) that leaves invariant the action functional to within a divergence yields a conservation law [9]-[12]. The Noether theorem has been the only general device allowing one, in the class of Euler-Lagrange equations, to reduce the search for conservation laws to a search for symmetries. In the last few years, effective methods have been devised for finding conservation laws for the very special class of so-called Lax equations. In 2000, Kara [13] presented the direct relationship between the conserved vector of a PDE and the Lie-Bäcklund symmetry generators of the PDE, from which it is possible for us to obtain conservation laws from symmetries (see, e.g. [14]).

The Lin-Tsien equation [15]

$$2u_{tx} + u_x u_{xx} - u_{yy} = 0, \quad (1)$$

where the variable  $u \equiv u(x, y, t)$  is a velocity potential, is known as a completely integrable model. It has been widely used to study dynamic transonic flow in two space dimensions, plasma physics, optics, condense matter physics, etc.

This paper is organized as follows. In section two, we derive the symmetry group theorem to the Lin-Tsien equation by using the modified CK's direct method, then the corresponding Lie point symmetry groups and infinite dimensional Kac-Moody-Virasoro symmetry algebra [16] are obtained straightforwardly. As a comparison, we also derive the Lie point symmetry groups by traditional Lie approaches and the result shows that both methods produce the same result. In section three, we first review some basic notions about Lie-Bäcklund operators and then using it we finally obtain conservation laws of the infinite dimensional Kac-Moody-Virasoro symmetry algebra as Lin-Tsien equation possessed up to second order group invariants. It is emphasized that equations with the same symmetries may possess the same types of conservation laws. The last section is a short summary and discussion.

## II. TRANSFORMATION GROUP BY THE DIRECT METHOD AND KAC-MOODY VIRASORO STRUCTURE OF THE LIE POINT SYMMETRY ALGEBRA

To find complete point symmetry transformation group of (1), one should find the general transformations in the following form

$$u = U(x, y, t, F(\xi, \eta, \tau)), \quad (2)$$

where  $\xi$ ,  $\eta$  and  $\tau$  are functions of  $x$ ,  $y$ ,  $t$ ,  $F \equiv F(\xi, \eta, \tau)$  is also solutions of the Lin-Tsien equation in the variables  $\xi, \eta$  and  $\tau$ , i.e.,

$$2F_{\tau, \xi} + F_{\xi}F_{\xi\xi} - F_{\eta\eta} = 0, \quad (3)$$

Fortunately, we can prove that for the Lin-Tsien equation it is enough to take

$$u = \alpha + \beta F(\xi, \eta, \tau), \quad (4)$$

instead of (2), where  $\alpha$ ,  $\beta$  and  $\xi$ ,  $\eta$ ,  $\tau$  are functions of  $\{x, y, t\}$ .

To prove the conclusion (4), one should submit the general expression (2) to Eq. (1). After eliminating  $F_{\eta\eta}$  and their higher derivatives via (3) and vanishing all the coefficients of the different terms of the derivatives of the functions  $F$ , one can get many complicated determining equations for 4 functions  $U \equiv U(x, y, t, F(\xi, \eta, \tau))$ ,  $\xi$ ,  $\eta$  and  $\tau$ . Two of them read as

$$\xi_x \eta_x^2 U_F F_{\xi} = 0, \quad \tau_x \xi_x^2 U_F F_{\tau} = 0,$$

For the reason that  $U_F$  should not be zero and there is no nontrivial solution for  $\xi_x = 0$ , the only way to cause the coefficients of  $F_\xi$  and  $F_\tau$  to vanish is

$$\eta_x = 0, \quad \tau_x = 0. \quad (5)$$

Under the condition (5), the determining equation containing  $F_\xi$  is

$$(U_F \xi_x^3 - \eta_y^2) F_\xi = 0.$$

solving the above equation for  $U$  then proves the conclusion that assumption (4) instead of the general one (2) is sufficient to find the general symmetry group of the Lin-Tsien equation.

Now the substitution of (4) with (5) into the Lin-Tsien equation leads to

$$\begin{aligned} & (-2\beta_y \eta_y + 2\beta_x \eta_t - \beta \eta_{yy}) F_\eta + \xi_x \beta (\xi_{xx} \beta + 2\xi_x \beta_x) F_\xi^2 + (\beta (\xi_x^3 \beta + \tau_y \xi_y \\ & - \tau_t \xi_x) F_{\xi\xi} + (\beta_x \beta \xi_{xx} + 2\beta_x^2 \xi_x + \beta \xi_x \beta_{xx}) F + 2\beta \xi_{tx} + \alpha_x \beta \xi_{xx} \\ & + 2\alpha_x \beta_x \xi_x - 2\beta_y \xi_y + \beta \xi_x \alpha_{xx} + 2\beta_t \xi_x + 2\beta_x \xi_t - \beta \xi_{yy}) F_\xi - \beta \tau_y^2 F_{\tau\tau} \\ & + (2\beta_x \tau_t - 2\beta_y \tau_y - \beta \tau_{yy}) F_\tau + (\beta_x \beta \xi_x^2 F + \beta (\alpha_x \xi_x^2 + 2\xi_t \xi_x - \xi_y^2)) F_{\xi\xi} \\ & + \beta (-\eta_y^2 + \tau_t \xi_x - \tau_y \xi_y) F_{\eta\eta} + 2\beta (\eta_t \xi_x - \eta_y \xi_y) F_{\xi\eta} + \beta_x \beta_{xx} F^2 \\ & + (2\beta_{tx} + \beta_x \alpha_{xx} + \alpha_x \beta_{xx} - \beta_{yy}) F + 2\alpha_{tx} + \alpha_x \alpha_{xx} - \alpha_{yy} \\ & - 2\beta \eta_y \tau_y F_{\tau\eta} = 0. \end{aligned} \quad (6)$$

Eq. (6) is true for arbitrary solutions  $F$  only when all the coefficients of the polynomials of the derivatives of  $F$  are zero which lead to a system of the determining equations for  $\xi$ ,  $\eta$ ,  $\tau$ ,  $\alpha$  and  $\beta$

$$-2\beta_y \eta_y + 2\beta_x \eta_t - \beta \eta_{yy} = 0, \quad \xi_x \beta (\xi_{xx} \beta + 2\xi_x \beta_x) = 0, \quad (7)$$

$$2\beta \xi_{tx} + \alpha_x \beta \xi_{xx} + 2\alpha_x \beta_x \xi_x - 2\beta_y \xi_y + \beta \xi_x \alpha_{xx} + 2\beta_t \xi_x + 2\beta_x \xi_t - \beta \xi_{yy} = 0, \quad (8)$$

$$\beta_x \beta \xi_{xx} + 2\beta_x^2 \xi_x + \beta \xi_x \beta_{xx} = 0, \quad \beta \tau_y^2 = 0, \quad 2\beta_x \tau_t - 2\beta_y \tau_y - \beta \tau_{yy} = 0, \quad (9)$$

$$(2\beta_{tx} + \beta_x \alpha_{xx} + \alpha_x \beta_{xx} - \beta_{yy}) = 0, \quad \beta (-\eta_y^2 + \tau_t \xi_x - \tau_y \xi_y) = 0, \quad (10)$$

$$\beta (\xi_x^3 \beta + \tau_y \xi_y - \tau_t \xi_x) = 0, \quad \beta_x \beta \xi_x^2 = 0, \quad 2\beta (\eta_t \xi_x - \eta_y \xi_y) = 0, \quad (11)$$

$$\beta_x \beta_{xx} = 0, \quad \beta(\alpha_x \xi_x^2 + 2\xi_t \xi_x - \xi_y^2) = 0, \quad 2\alpha_{tx} + \alpha_x \alpha_{xx} - \alpha_{yy} = 0, \quad (12)$$

$$\beta \eta_y \tau_y = 0. \quad (13)$$

It is straightforward to obtain the general solutions of the determining equations (7)–(13).

The results are

$$\xi = \tau_t^{\frac{1}{3}} x + \frac{1}{3} \frac{y^2 \tau_{tt}}{\tau_t^{\frac{2}{3}}} + \frac{y \eta_{0t}}{\tau_t^{\frac{1}{3}}} + \xi_0, \eta = \tau_t^{\frac{2}{3}} y + \eta_0, \beta = \tau_t^{\frac{1}{3}}, \quad (14)$$

$$\begin{aligned} \alpha = & -\frac{(-36\tau_{tt}\tau_{ttt}\tau_t + 28\tau_{tt}^3 + 9\tau_{ttt}\tau_t^2)y^4}{81\tau_t^3} \\ & -\frac{2(-2\eta_{0tt}\tau_{tt}\tau_t + \eta_{0ttt}\tau_t^2 - \tau_{ttt}\eta_{0t}\tau_t + 2\tau_{tt}^2\eta_{0t})y^3}{3\tau_t^{\frac{8}{3}}} \\ & +(-\frac{2(-4\tau_{tt}^2 + 3\tau_{ttt}\tau_t)x}{9\tau_t^2} - \frac{-4\tau_{tt}\xi_{0t}\tau_t - 6\eta_{0t}\eta_{0tt}\tau_t + 6\xi_{0tt}\tau_t^2 + 5\tau_{tt}\eta_{0t}^2}{3\tau_t^{\frac{7}{3}}})y^2 \\ & +(-\frac{2(-\eta_{0t}\tau_{tt} + \eta_{0tt}\tau_t)x}{\tau_t^{\frac{5}{3}}} + \alpha_2)y - \frac{\tau_{tt}x^2}{3\tau_t} - \frac{(2\xi_{0t}\tau_t - \eta_{0t}^2)x}{\tau_t^{\frac{4}{3}}} + \alpha_1 \end{aligned} \quad (15)$$

where  $\xi_0 \equiv \xi_0(t)$ ,  $\eta_0 \equiv \eta_0(t)$ ,  $\tau \equiv \tau(t)$ ,  $\alpha_1 \equiv \alpha_1(t)$  and  $\alpha_2 \equiv \alpha_2(t)$  are arbitrary functions of time  $t$ .

In summary, the following theorem holds:

**Theorem 1:** If  $F = F(x, y, t)$  is a solution of the Lin-Tsien equation (1), then so is

$$\begin{aligned} u = & -\frac{(-36\tau_{tt}\tau_{ttt}\tau_t + 28\tau_{tt}^3 + 9\tau_{ttt}\tau_t^2)y^4}{81\tau_t^3} \\ & -\frac{2(-2\eta_{0tt}\tau_{tt}\tau_t + \eta_{0ttt}\tau_t^2 - \tau_{ttt}\eta_{0t}\tau_t + 2\tau_{tt}^2\eta_{0t})y^3}{3\tau_t^{\frac{8}{3}}} \\ & +(-\frac{2(-4\tau_{tt}^2 + 3\tau_{ttt}\tau_t)x}{9\tau_t^2} - \frac{-4\tau_{tt}\xi_{0t}\tau_t - 6\eta_{0t}\eta_{0tt}\tau_t + 6\xi_{0tt}\tau_t^2 + 5\tau_{tt}\eta_{0t}^2}{3\tau_t^{\frac{7}{3}}})y^2 \\ & +(-\frac{2(-\eta_{0t}\tau_{tt} + \eta_{0tt}\tau_t)x}{\tau_t^{\frac{5}{3}}} + \alpha_2)y - \frac{\tau_{tt}x^2}{3\tau_t} - \frac{(2\xi_{0t}\tau_t - \eta_{0t}^2)x}{\tau_t^{\frac{4}{3}}} \\ & +\alpha_1 + \tau_t^{\frac{1}{3}}F(\xi, \eta, \tau), \end{aligned} \quad (16)$$

with (14), where  $\xi_0$ ,  $\eta_0$ ,  $\tau$ ,  $\alpha_1$  and  $\alpha_2$  are arbitrary functions of  $t$ .

Applying the theorem to some simple exact solutions without arbitrary functions, one may obtain some types of novel generalized solutions with some arbitrary functions. In the following, we just present one special solution example.

**Example 1.** It is quite trivial that the Lin-Tsien equation (1) possesses a special simple solution

$$F = 1. \quad (17)$$

Using the transformation theorem to the above special solution we have the following new special solution of the Lin-Tsien equation

$$\begin{aligned} u = & -\frac{(-36\tau_{tt}\tau_{ttt}\tau_t + 28\tau_{tt}^3 + 9\tau_{ttt}\tau_t^2)y^4}{81\tau_t^3} \\ & -\frac{2(-2\eta_{0t}\tau_{tt}\tau_t + \eta_{0tt}\tau_t^2 - \tau_{tt}\eta_{0t}\tau_t + 2\tau_{tt}^2\eta_{0t})y^3}{3\tau_t^{\frac{8}{3}}} \\ & +\left(-\frac{2(-4\tau_{tt}^2 + 3\tau_{ttt}\tau_t)x}{9\tau_t^2} - \frac{-4\tau_{tt}\xi_{0t}\tau_t - 6\eta_{0t}\eta_{0tt}\tau_t + 6\xi_{0tt}\tau_t^2 + 5\tau_{tt}\eta_{0t}^2}{3\tau_t^{\frac{7}{3}}}\right)y^2 \\ & +\left(-\frac{2(-\eta_{0t}\tau_{tt} + \eta_{0tt}\tau_t)x}{\tau_t^{\frac{5}{3}}} + \alpha_2\right)y - \frac{\tau_{tt}x^2}{3\tau_t} - \frac{(2\xi_{0t}\tau_t - \eta_{0t}^2)x}{\tau_t^{\frac{4}{3}}} \\ & +\alpha_1 + \tau_t^{\frac{1}{3}}. \end{aligned} \quad (18)$$

In the traditional Lie group theory, one always tries to find the Lie point symmetries first and then use the Lie's first fundamental theorem to obtain the symmetry transformation group. Conversely, we are fortunate to obtain the symmetry transformation group in the first place by a simple direct method. Once the transformation group is known, the Lie point symmetries and the related Lie symmetry algebra can be obtained straightforward by a more simple limiting procedure.

For the Lin-Tsien (1), the corresponding Lie point symmetries can be derived from the symmetry group transformation theorem by setting

$$\begin{aligned} \eta_0(t) &= \epsilon h(t), \quad \tau(t) = t + \epsilon f(t), \quad \xi_0(t) = \epsilon g(t), \quad \alpha_1(t) = \epsilon m(t), \\ \alpha_2(t) &= \epsilon n(t), \end{aligned} \quad (19)$$

with  $\epsilon$  being an infinitesimal parameter, then (16) can be written as

$$\begin{aligned} u &= F + \epsilon\sigma(F) + O(\epsilon^2), \\ \sigma(F) &= (g(t) + \frac{1}{3}f_t(t)x + \frac{1}{3}f_{tt}(t)y^2 + yh_t(t))F_x + (h(t) + \frac{2}{3}f_t(t)y)F_y \\ &\quad + fF_t + \frac{1}{3}f_t(t)F - \frac{2}{3}f_{tt}xy^2 - \frac{1}{3}f_{tt}x^2 - 2g_t(t)x + m(t) \\ &\quad + n(t)y - \frac{1}{9}f_{ttt}y^4 - \frac{2}{3}h_{tt}(t)y^3 - 2g_{tt}y^2 - 2h_{tt}xy, \end{aligned} \quad (20)$$

The equivalent vector expression of the above symmetry reads

$$\begin{aligned}
V = & \left\{ \left( \frac{1}{3} f_t(t)x + \frac{1}{3} f_{tt}(t)y^2 \right) \frac{\partial}{\partial x} + \frac{2}{3} f_t(t)y \frac{\partial}{\partial y} + f(t) \frac{\partial}{\partial t} \right. \\
& - \left( \frac{1}{3} f_t(t)F - \frac{2}{3} f_{ttt}xy^2 - \frac{1}{3} f_{tt}x^2 - \frac{1}{9} f_{ttt}y^4 \right) \frac{\partial}{\partial F} \Big\} \\
& + \left\{ g(t) \frac{\partial}{\partial x} + (2g_t(t)x + 2g_{tt}y^2) \frac{\partial}{\partial F} \right\} + \left\{ y h_t(t) \frac{\partial}{\partial x} + h(t) \frac{\partial}{\partial y} \right. \\
& + \left( \frac{2}{3} h_{ttt}(t)y^3 + 2h_{tt}xy \right) \frac{\partial}{\partial F} \Big\} - \left\{ m(t) \frac{\partial}{\partial F} \right\} - \left\{ n(t)y \frac{\partial}{\partial F} \right\} \\
& \equiv V_1(f(t)) + V_2(g(t)) + V_3(h(t)) + V_4(m(t)) + V_5(n(t)).
\end{aligned} \tag{21}$$

Since the functions  $f$ ,  $g$ ,  $h$ ,  $m$  and  $n$  are arbitrary, the corresponding Lie algebra is an infinite dimensional Lie algebra.

It is easy to verify that the symmetries  $V_i$ ,  $i = 1, 2, 3, 4, 5$  constitute an infinite dimensional Kac-Moody-Virasoro [16] type symmetry algebra  $\mathcal{S}$  with the following nonzero commutation relations

$$[V_1(f), V_4(m)] = V_4\left(\frac{1}{3}mf_t + fm_t\right), \tag{22}$$

$$[V_2(g), V_3(h)] = V_5(2(2hg_{tt} - gh_{tt} + h_tg_t)), \tag{23}$$

$$[V_1(f), V_3(h)] = V_3\left(fh_t - \frac{2}{3}hf_t\right), \tag{24}$$

$$[V_1(f), V_2(g)] = V_2\left(g_tf - \frac{1}{3}gft\right), \tag{25}$$

$$[V_2(g_1), V_2(g_2)] = V_4(2g_{1t}g_2 - 2g_1g_{2t}), \tag{26}$$

$$[V_3(h), V_5(n)] = V_4(hn), \tag{27}$$

$$[V_1(f), V_5(n)] = V_5(f_tn + fn_t), \tag{28}$$

$$[V_1(f_1), V_1(f_2)] = V_1(f_1f_{2t} - f_2f_{1t}), \tag{29}$$

$$[V_3(h_1), V_3(h_2)] = V_2(h_1h_{2t} - h_2h_{1t}), \tag{30}$$

It should be emphasized that the algebra is infinite dimensional because the generators  $V_1$ ,  $V_2$ ,  $V_3$ ,  $V_4$  and  $V_5$  all contain *arbitrary* functions. The algebra is closed because all the commutators can be expressed by the generators belong to the generator set usually with different functions and the generators contained *different* functions belong to the set. Especially, it is clear that the symmetry  $V_1(f)$  constitute an centerless Virasoro symmetry algebra.

As a comparison, we now derive the Lie point symmetry of the Lin-Tsien equation by the standard Lie approach briefly.

To study the symmetry of the equation (1), we search for the Lie point symmetry transformations in the vector form

$$V = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u},$$

where  $X$ ,  $Y$ ,  $T$  and  $U$  are functions with respect to  $x$ ,  $y$ ,  $t$ ,  $u$ , which means that (1) is invariant under the point transformation

$$\{x, y, t, u\} \rightarrow \{x + \epsilon X, y + \epsilon Y, t + \epsilon T, u + \epsilon U\}$$

with infinitesimal parameter  $\epsilon$ .

In other words, the symmetry of the equation (1) can be written as the function form

$$\sigma = Xu_x + Yu_y + Tu_t - U, \quad (31)$$

where the symmetry  $\sigma$  is a solution of the linearized equation for (1)

$$2\sigma_{tx} + \sigma_x u_{xx} + u_x \sigma_{xx} - \sigma_{yy} = 0, \quad (32)$$

which is obtained by substituting  $u = u + \epsilon \sigma$  into (1) and dropping the nonlinear terms in  $\sigma$ .

It is easy to solve out  $X(x, y, t, u)$ ,  $Y(x, y, t, u)$ ,  $T(x, y, t, u)$  and  $U(x, y, t, u)$  by substituting (31) into (32), and eliminating  $u_{yy}$  and its higher order derivatives by means of the Lin-Tsien equation. After taking the constants as zero, we get the results

$$X(x, y, t, u) = \frac{1}{3}T_t x + \frac{1}{3}T_{tt}y^2 + X_t y + Y, \quad (33)$$

$$Y(x, y, t, u) = \frac{2}{3}T_t y + X, \quad (34)$$

$$T(x, y, t, u) = T(t), \quad (35)$$

$$\begin{aligned} U(x, y, t, u) = & \frac{1}{3}x^2 T_{tt} - \frac{1}{3}u T_t + \frac{2}{3}xy^2 T_{ttt} + 2x X_{tt}y + 2x Y_t \\ & + \frac{1}{9}y^4 T_{ttt} + \frac{2}{3}X_{ttt}y^3 + 2Y_{tt}y^2 + Z_1 y + Z_2, \end{aligned} \quad (36)$$

where  $X$ ,  $Y$ ,  $T$ ,  $Z_1$  and  $Z_2$  are arbitrary functions of  $t$ .



The vector form of the Lie point symmetries reads

$$\begin{aligned}
V = & \left( \frac{1}{3}T_t + \frac{1}{3}T_{tt}y^2 + X_ty + Y \right) \frac{\partial}{\partial x} + \left( \frac{2}{3}T_t + X \right) \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} \\
& + \left( \frac{1}{3}uT_t - \frac{1}{3}x^2T_{tt} - \frac{2}{3}xy^2T_{ttt} - 2xX_{tt}y - 2xY_t - \frac{1}{9}y^4T_{ttt} \right. \\
& \left. - \frac{2}{3}X_{ttt}y^3 - 2Y_{tt}y^2 - Z_1y - Z_2 \right) \frac{\partial}{\partial u},
\end{aligned} \tag{37}$$

which is exactly the same as that obtained by the modified CK's approach.

### III. CONSERVATION LAWS RELATED TO THE THE SYMMETRY (37)

In order to obtain conservation laws related to the symmetry (37), we need some basic notions about Lie-Bäcklund operators first.

A Lie-Bäcklund operator is given by

$$X_0 = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} + \zeta_{i_1 i_2} \frac{\partial}{\partial u_{i_1 i_2}} + \cdots, \tag{38}$$

where  $\xi^i$ ,  $\eta$  and the additional coefficients are

$$\begin{aligned}
\zeta_i &= D_i(W) + \xi^j u_{ij}, \\
\zeta_{i_1 i_2} &= D_{i_1 i_2}(W) + \xi^j u_{ji_1 i_2},
\end{aligned} \tag{39}$$

and  $W$  is the Lie characteristic function defined by

$$W = \eta - \xi^j u_j \tag{40}$$

with  $D_i$  being the operator of total differentiation

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \quad i = 1, \dots, n, \tag{41}$$

as

$$u_i = D_i(u), \quad u_{ij} = D_j D_i(u). \tag{42}$$

These definitions and results relating to Lie-Bäcklund operator can be found in [17] and the repeated indices mean the summations known as the Einstein summation rule.

Using equations (38)-(42), we can calculate the 2nd-order Lie-Bäcklund operator of the vector field  $V$  defined by equation (37):

$$\xi^x = \frac{1}{3}xT_t + \frac{1}{3}T_{tt}y^2 + X_ty + Y, \tag{43}$$

$$\xi^y = \frac{2}{3}yT_t + X, \quad (44)$$

$$\xi^t = T, \quad (45)$$

$$\begin{aligned} \eta = & -\frac{1}{3}uT_t + \frac{1}{3}x^2T_{tt} + \frac{2}{3}xy^2T_{ttt} + 2xX_{tt}y + 2xY_t + \frac{1}{9}y^4T_{ttt} \\ & + \frac{2}{3}X_{ttt}y^3 + 2Y_{tt}y^2 + Z_1y + Z_2, \end{aligned} \quad (46)$$

$$\zeta_x = -\frac{2}{3}T_tu_x + \frac{2}{3}xT_{tt} + \frac{2}{3}y^2T_{ttt} + 2X_{tt}y + 2Y_t, \quad (47)$$

$$\begin{aligned} \zeta_y = & -T_tu_y + \frac{4}{3}T_{ttt}yx + 2xX_{tt} + \frac{4}{9}y^3T_{ttt} + 2X_{ttt}y^2 + 4Y_{tt}y \\ & + Z_1 - \left(\frac{2}{3}T_{tt}y + X_t\right)u_x, \end{aligned} \quad (48)$$

$$\begin{aligned} \zeta_t = & -\frac{4}{3}T_tu_t - \frac{1}{3}uT_{tt} + \frac{1}{3}x^2T_{ttt} + \frac{2}{3}xy^2T_{ttt} + 2xyX_{ttt} + 2xY_{tt} \\ & + \frac{1}{9}y^4T_{ttt} + \frac{2}{3}X_{ttt}y^3 + 2Y_{ttt}y^2 + Z_{1,t}y + Z_{2,t} \\ & - \left(\frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + yX_{tt} + Y_t\right)u_x - \left(\frac{2}{3}yT_{tt} + X_t\right)u_y, \end{aligned} \quad (49)$$

$$\zeta_{xx} = -u_{xx}T_t + \frac{2}{3}T_{tt}, \quad (50)$$

$$\zeta_{xy} = -\frac{4}{3}T_tu_{xy} + \frac{4}{3}yT_{ttt} + 2X_{tt} - \left(\frac{2}{3}yT_{tt} + X_t\right)u_{xx}, \quad (51)$$

$$\begin{aligned} \zeta_{xt} = & -\frac{5}{3}T_tu_{xt} - \frac{2}{3}T_{tt}u_x + \frac{2}{3}xT_{ttt} + \frac{2}{3}y^2T_{ttt} + 2X_{ttt}y + 2Y_{tt} \\ & - \left(\frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + yX_{tt} + Y_t\right)u_{xx} - \left(\frac{2}{3}T_{tt}y + X_t\right)u_{xy}, \end{aligned} \quad (52)$$

$$\begin{aligned} \zeta_{yy} = & -\frac{5}{3}T_tu_{yy} + \frac{4}{3}xT_{ttt} + \frac{4}{3}y^2T_{ttt} + 4X_{ttt}y + 4Y_{tt} - \frac{2}{3}u_xT_{tt} \\ & - 2\left(\frac{2}{3}yT_{tt} + X_t\right)u_{xy}, \end{aligned} \quad (53)$$

$$\begin{aligned} \zeta_{yt} = & -2T_tu_{yt} - T_{tt}u_y + \frac{4}{3}xyT_{ttt} + 2xX_{ttt} + \frac{4}{9}y^3T_{ttt} + 2X_{ttt}y^2 \\ & + 4yY_{ttt} + Z_{1,t} - \left(\frac{2}{3}yT_{ttt} + X_{tt}\right)u_x - \left(\frac{2}{3}yT_{tt} + X_t\right)u_{xt} \\ & - \left(\frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + yX_{tt} + Y_t\right)u_{xy} - \left(\frac{2}{3}yT_{tt} + X_t\right)u_{yy}, \end{aligned} \quad (54)$$

$$\begin{aligned}
\zeta_{tt} = & -\frac{7}{3}T_t u_{tt} - \frac{5}{3}T_{tt} u_t - \frac{1}{3}T_{ttt} u + \frac{1}{3}x^2 T_{tttt} + \frac{2}{3}xy^2 T_{tttt} + 2xy X_{ttt} \\
& + 2xY_{ttt} + \frac{1}{9}y^4 T_{ttttt} + \frac{2}{3}X_{tttt} y^3 + 2Y_{tttt} y^2 + Z_{1,tt} y + Z_{2,tt} \\
& - \left(\frac{1}{3}x T_{ttt} + \frac{1}{3}y^2 T_{ttt} + y X_{ttt} + Y_{tt}\right) u_x - 2\left(\frac{1}{3}x T_{tt} + \frac{1}{3}y^2 T_{tt}\right. \\
& \left. + y X_{tt} + Y_t\right) u_{xt} - \left(\frac{2}{3}y T_{ttt} + X_{tt}\right) u_y - 2\left(\frac{2}{3}y T_{tt} + X_t\right) u_{yt}.
\end{aligned} \tag{55}$$

Correspondingly, the second order Lie-Bäcklund operator is given by

$$\begin{aligned}
X_0 = & \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \xi^t \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_u \frac{\partial}{\partial u_{xx}} \\
& + \zeta_{xy} \frac{\partial}{\partial u_{xy}} + \zeta_{xt} \frac{\partial}{\partial u_{xt}} + \zeta_{yy} \frac{\partial}{\partial u_{yy}} + \zeta_{yt} \frac{\partial}{\partial u_{yt}} + \zeta_{tt} \frac{\partial}{\partial u_{tt}}.
\end{aligned} \tag{56}$$

**Theorem**([18], [13]): Suppose that  $X_0$  is a Lie-Bäcklund symmetry of (1) such that the conservation vector  $T' = (T_1, T_2, T_3)$  is invariant under  $X_0$ . Then

$$X_0(T_i) + \sum_{j=1}^3 T_i D_j \xi_j - \sum_{j=1}^3 T_j D_j (\xi_i) = 0, \quad (i = 1, 2, 3), \tag{57}$$

where  $D_1 = D_x$ ,  $D_2 = D_y$ ,  $D_3 = D_t$  and  $\xi_i$  are determined by (56).

A Lie-Bäcklund symmetry  $X_0$  is said to be associated with a conserved vector  $T'$  of (1) if  $X_0$  and  $T'$  satisfy relations (57).

Now we construct the corresponding conservation laws relating to (37) in the form

$$D_x J_1 + D_y J_2 + D_t \rho = 0, \tag{58}$$

where  $T_1 = J_1$ ,  $T_2 = J_2$ ,  $T_3 = \rho$  with  $J_1$ ,  $J_2$  and  $\rho$  being functions of  $\{x, y, t, u, u_x, u_y, \dots, u_{tt}\}$ .

In terms of  $T' = (J_1, J_2, \rho)$ , Eq. (57) is equivalent to the following three equations:

$$\begin{aligned}
& \left( \frac{1}{3}xT_t + \frac{1}{3}y^2T_{tt} + yX_t + Y \right) \frac{\partial J_1}{\partial x} + \left( \frac{2}{3}yT_t + X \right) \frac{\partial J_1}{\partial y} + T \frac{\partial J_1}{\partial t} + \left( -\frac{1}{3}uT_t \right. \\
& + \frac{1}{3}x^2T_{tt} + \frac{2}{3}xy^2T_{ttt} + 2xyX_{tt} + 2xY_t + \frac{1}{9}y^4T_{tttt} + \frac{2}{3}X_{ttt}y^3 + 2Y_{tt}y^2 \\
& + Z_1y + Z_2 \left. \right) \frac{\partial J_1}{\partial u} + \left( -\frac{2}{3}T_tu_x + \frac{2}{3}xT_{tt} + \frac{2}{3}y^2T_{ttt} + 2X_{tt}y + 2Y_t \right) \frac{\partial J_1}{\partial u_x} \\
& + \left[ -T_tu_y + \frac{4}{3}xyT_{ttt} + 2xX_{tt} + \frac{4}{9}y^3T_{tttt} + 2X_{ttt}y^2 + 4Y_{tt}y + Z_1 \right. \\
& \left. - \left( \frac{2}{3}yT_{tt} + X_t \right) u_x \right] \frac{\partial J_1}{\partial u_y} + \left[ -\frac{4}{3}T_tu_t - \frac{1}{3}uT_{tt} + \frac{1}{3}x^2T_{ttt} + \frac{2}{3}xy^2T_{ttt} \right. \\
& + 2xyX_{ttt} + 2xY_{tt} + \frac{1}{9}y^4T_{tttt} + \frac{2}{3}X_{ttt}y^3 + 2Y_{ttt}y^2 + Z_{1,t}y + Z_{2,t} \\
& \left. - \left( \frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + yX_{tt} + Y_t \right) u_x - \left( \frac{2}{3}yT_{tt} + X_t \right) u_y \right] \frac{\partial J_1}{\partial u_t} + \left( -T_tu_{xx} \right. \\
& + \frac{2}{3}T_{tt} \left. \right) \frac{\partial J_1}{\partial u_{xx}} + \left[ -\frac{4}{3}T_tu_{xy} + \frac{4}{3}yT_{ttt} + 2X_{tt} - \left( \frac{2}{3}yT_{tt} + X_t \right) u_{xx} \right] \frac{\partial J_1}{\partial u_{xy}} \\
& + \left[ -\frac{5}{3}T_tu_{xt} - \frac{2}{3}T_{tt}u_x + \frac{2}{3}xT_{ttt} + \frac{2}{3}y^2T_{ttt} + 2X_{ttt}y + 2Y_{tt} - \left( \frac{1}{3}xT_{tt} \right. \right. \\
& + \frac{1}{3}y^2T_{ttt} + yX_{tt} + Y_t \left. \right) u_{xx} - \left( \frac{2}{3}yT_{tt} + X_t \right) u_{xy} \left. \right] \frac{\partial J_1}{\partial u_{xt}} + \left[ -\frac{5}{3}T_tu_{yy} + \frac{4}{3}xT_{ttt} \right. \\
& + \frac{4}{3}y^2T_{ttt} + 4X_{ttt}y + 4Y_{tt} - \frac{2}{3}u_xT_{tt} - 2\left( \frac{2}{3}yT_{tt} + X_t \right) u_{xy} \left. \right] \frac{\partial J_1}{\partial u_{yy}} \\
& + \left[ -2u_{yt}T_t - u_yT_{tt} + \frac{4}{3}xyT_{ttt} + 2xX_{ttt} + \frac{4}{9}y^3T_{tttt} + 2X_{ttt}y^2 + 4Y_{ttt}y \right. \\
& + Z_{1,t} - \left( \frac{2}{3}yT_{ttt} + X_{tt} \right) u_x - \left( \frac{2}{3}yT_{tt} + X_t \right) u_{tx} - \left( \frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + X_{tt}y \right. \\
& + Y_t \left. \right) u_{xy} - \left( \frac{2}{3}yT_{tt} + X_t \right) u_{yy} \left. \right] \frac{\partial J_1}{\partial u_{yt}} + \left[ -\frac{7}{3}u_{tt}T_t - \frac{5}{3}u_tT_{tt} - \frac{1}{3}uT_{ttt} \right. \\
& + \frac{1}{3}x^2T_{ttt} + \frac{2}{3}xy^2T_{tttt} + 2xX_{ttt}y + 2xY_{ttt} + \frac{1}{9}y^4T_{ttttt} + \frac{2}{3}X_{tttt}y^3 + 2Y_{tttt}y^2 \\
& + Z_{1,tt}y + Z_{2,tt} - \left( \frac{1}{3}xT_{ttt} + \frac{1}{3}y^2T_{ttt} + X_{ttt}y + Y_{tt} \right) u_x - 2\left( \frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} \right. \\
& + X_{tt}y + Y_t \left. \right) u_{tx} - \left( \frac{2}{3}yT_{ttt} + X_{tt} \right) u_y - 2\left( \frac{2}{3}yT_{tt} + X_t \right) u_{yt} \left. \right] \frac{\partial J_1}{\partial u_{tt}} + \frac{5}{3}T_tJ_1 \\
& - \left( X_t + \frac{2}{3}yT_{tt} \right) J_2 - \left( \frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + yX_{tt} + Y_t \right) \rho = 0,
\end{aligned} \tag{59}$$

$$\begin{aligned}
& \left( \frac{1}{3}xT_t + \frac{1}{3}y^2T_{tt} + yX_t + Y \right) \frac{\partial J_2}{\partial x} + \left( \frac{2}{3}yT_t + X \right) \frac{\partial J_2}{\partial y} + T \frac{\partial J_2}{\partial t} + \left( -\frac{1}{3}uT_t \right. \\
& + \frac{1}{3}x^2T_{tt} + \frac{2}{3}xy^2T_{ttt} + 2xyX_{tt} + 2xY_t + \frac{1}{9}y^4T_{ttt} + \frac{2}{3}X_{ttt}y^3 + 2Y_{tt}y^2 \\
& + Z_1y + Z_2 \left. \right) \frac{\partial J_2}{\partial u} + \left( -\frac{2}{3}T_tu_x + \frac{2}{3}xT_{tt} + \frac{2}{3}y^2T_{ttt} + 2X_{tt}y + 2Y_t \right) \frac{\partial J_2}{\partial u_x} \\
& + \left[ -T_tu_y + \frac{4}{3}xyT_{ttt} + 2xX_{tt} + \frac{4}{9}y^3T_{ttt} + 2X_{ttt}y^2 + 4Y_{tt}y + Z_1 \right. \\
& \left. - \left( \frac{2}{3}yT_{tt} + X_t \right) u_x \right] \frac{\partial J_2}{\partial u_y} + \left[ -\frac{4}{3}T_tu_t - \frac{1}{3}uT_{tt} + \frac{1}{3}x^2T_{ttt} + \frac{2}{3}xy^2T_{ttt} \right. \\
& + 2xyX_{ttt} + 2xY_{tt} + \frac{1}{9}y^4T_{tttt} + \frac{2}{3}X_{ttt}y^3 + 2Y_{ttt}y^2 + Z_{1,t}y + Z_{2,t} \\
& \left. - \left( \frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + yX_{tt} + Y_t \right) u_x - \left( \frac{2}{3}yT_{tt} + X_t \right) u_y \right] \frac{\partial J_2}{\partial u_t} + (-T_tu_{xx} \\
& + \frac{2}{3}T_{tt}) \frac{\partial J_2}{\partial u_{xx}} + \left[ -\frac{4}{3}T_tu_{xy} + \frac{4}{3}yT_{ttt} + 2X_{tt} - \left( \frac{2}{3}yT_{tt} + X_t \right) u_{xx} \right] \frac{\partial J_2}{\partial u_{xy}} \\
& + \left[ -\frac{5}{3}T_tu_{xt} - \frac{2}{3}T_{tt}u_x + \frac{2}{3}xT_{ttt} + \frac{2}{3}y^2T_{ttt} + 2X_{ttt}y + 2Y_{tt} - \left( \frac{1}{3}xT_{tt} \right. \right. \\
& + \frac{1}{3}y^2T_{ttt} + yX_{tt} + Y_t) u_{xx} - \left. \left( \frac{2}{3}yT_{tt} + X_t \right) u_{xy} \right] \frac{\partial J_2}{\partial u_{xt}} + \left[ -\frac{5}{3}T_tu_{yy} \right. \\
& + \frac{4}{3}xT_{ttt} + \frac{4}{3}y^2T_{ttt} + 4X_{ttt}y + 4Y_{tt} - \frac{2}{3}u_xT_{tt} - 2\left( \frac{2}{3}yT_{tt} + X_t \right) u_{xy} \left. \right] \frac{\partial J_2}{\partial u_{yy}} \\
& + \left[ -2u_{yt}T_t - u_yT_{tt} + \frac{4}{3}xyT_{ttt} + 2xX_{ttt} + \frac{4}{9}y^3T_{ttt} + 2X_{ttt}y^2 + 4Y_{ttt}y + Z_{1,t} \right. \\
& - \left( \frac{2}{3}yT_{ttt} + X_{tt} \right) u_x - \left( \frac{2}{3}yT_{tt} + X_t \right) u_{tx} - \left( \frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + X_{tt}y + Y_t \right) u_{xy} \\
& \left. - \left( \frac{2}{3}yT_{tt} + X_t \right) u_{yy} \right] \frac{\partial J_2}{\partial u_{yt}} + \left[ -\frac{7}{3}u_{tt}T_t - \frac{5}{3}u_tT_{tt} - \frac{1}{3}uT_{ttt} + \frac{1}{3}x^2T_{ttt} \right. \\
& + \frac{2}{3}xy^2T_{ttt} + 2xX_{ttt}y + 2xY_{ttt} + \frac{1}{9}y^4T_{ttt} + \frac{2}{3}X_{ttt}y^3 + 2Y_{ttt}y^2 \\
& + Z_{1,tt}y + Z_{2,tt} - \left( \frac{1}{3}xT_{ttt} + \frac{1}{3}y^2T_{ttt} + X_{ttt}y + Y_{tt} \right) u_x - 2\left( \frac{1}{3}xT_{tt} \right. \\
& + \frac{1}{3}y^2T_{ttt} + X_{tt}y + Y_t) u_{tx} - \left. \left( \frac{2}{3}yT_{ttt} + X_{tt} \right) u_y - 2\left( \frac{2}{3}yT_{tt} + X_t \right) u_{yt} \right] \frac{\partial J_2}{\partial u_{tt}} \\
& + \frac{4}{3}T_tJ_2 - \left( \frac{2}{3}yT_{tt} + X_t \right) \rho = 0,
\end{aligned} \tag{60}$$

$$\begin{aligned}
& \left( \frac{1}{3}xT_t + \frac{1}{3}y^2T_{tt} + yX_t + Y \right) \frac{\partial \rho}{\partial x} + \left( \frac{2}{3}yT_t + X \right) \frac{\partial \rho}{\partial y} + T \frac{\partial \rho}{\partial t} + \left( -\frac{1}{3}uT_t \right. \\
& + \frac{1}{3}x^2T_{tt} + \frac{2}{3}xy^2T_{ttt} + 2xyX_{tt} + 2xY_t + \frac{1}{9}y^4T_{ttt} + \frac{2}{3}X_{ttt}y^3 + 2Y_{tt}y^2 \\
& + Z_1y + Z_2 \left. \right) \frac{\partial \rho}{\partial u} + \left( -\frac{2}{3}T_tu_x + \frac{2}{3}xT_{tt} + \frac{2}{3}y^2T_{ttt} + 2X_{tt}y + 2Y_t \right) \frac{\partial \rho}{\partial u_x} \\
& + \left[ -T_tu_y + \frac{4}{3}xyT_{ttt} + 2xX_{tt} + \frac{4}{9}y^3T_{ttt} + 2X_{ttt}y^2 + 4Y_{tt}y + Z_1 \right. \\
& \left. - \left( \frac{2}{3}yT_{tt} + X_t \right) u_x \right] \frac{\partial \rho}{\partial u_y} + \left[ -\frac{4}{3}T_tu_t - \frac{1}{3}uT_{tt} + \frac{1}{3}x^2T_{ttt} + \frac{2}{3}xy^2T_{ttt} \right. \\
& + 2xyX_{ttt} + 2xY_{tt} + \frac{1}{9}y^4T_{ttt} + \frac{2}{3}X_{ttt}y^3 + 2Y_{ttt}y^2 + Z_{1,t}y + Z_{2,t} \\
& \left. - \left( \frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + yX_{tt} + Y_t \right) u_x - \left( \frac{2}{3}yT_{tt} + X_t \right) u_y \right] \frac{\partial \rho}{\partial u_t} \\
& + \left( -T_tu_{xx} + \frac{2}{3}T_{tt} \right) \frac{\partial \rho}{\partial u_{xx}} + \left[ -\frac{4}{3}T_tu_{xy} + \frac{4}{3}yT_{ttt} + 2X_{tt} - \left( \frac{2}{3}yT_{tt} \right. \right. \\
& \left. \left. + X_t \right) u_{xx} \right] \frac{\partial \rho}{\partial u_{xy}} + \left[ -\frac{5}{3}T_tu_{xt} - \frac{2}{3}T_{tt}u_x + \frac{2}{3}xT_{ttt} + \frac{2}{3}y^2T_{ttt} + 2X_{ttt}y \right. \\
& + 2Y_{tt} - \left( \frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + yX_{tt} + Y_t \right) u_{xx} - \left( \frac{2}{3}yT_{tt} + X_t \right) u_{xy} \left. \right] \frac{\partial \rho}{\partial u_{xt}} \\
& + \left[ -\frac{5}{3}T_tu_{yy} + \frac{4}{3}xT_{ttt} + \frac{4}{3}y^2T_{ttt} + 4X_{ttt}y + 4Y_{tt} - \frac{2}{3}u_xT_{tt} \right. \\
& \left. - 2 \left( \frac{2}{3}yT_{tt} + X_t \right) u_{xy} \right] \frac{\partial \rho}{\partial u_{yy}} + \left[ -2u_{yt}T_t - u_yT_{tt} + \frac{4}{3}xyT_{ttt} + 2xX_{ttt} \right. \\
& + \frac{4}{9}y^3T_{ttt} + 2X_{ttt}y^2 + 4Y_{ttt}y + Z_{1,t} - \left( \frac{2}{3}yT_{ttt} + X_{tt} \right) u_x - \left( \frac{2}{3}yT_{tt} \right. \\
& \left. + X_t \right) u_{tx} - \left( \frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + X_{tt}y + Y_t \right) u_{xy} - \left( \frac{2}{3}yT_{tt} + X_t \right) u_{yy} \left. \right] \frac{\partial \rho}{\partial u_{yt}} \\
& + \left[ -\frac{7}{3}u_{tt}T_t - \frac{5}{3}u_tT_{tt} - \frac{1}{3}uT_{ttt} + \frac{1}{3}x^2T_{ttt} + \frac{2}{3}xy^2T_{ttt} + 2xX_{ttt}y + 2xY_{ttt} \right. \\
& + \frac{1}{9}y^4T_{ttt} + \frac{2}{3}X_{ttt}y^3 + 2Y_{ttt}y^2 + Z_{1,tt}y + Z_{2,tt} - \left( \frac{1}{3}xT_{ttt} + \frac{1}{3}y^2T_{ttt} \right. \\
& + X_{ttt}y + Y_{tt} \right) u_x - 2 \left( \frac{1}{3}xT_{tt} + \frac{1}{3}y^2T_{ttt} + X_{tt}y + Y_t \right) u_{tx} - \left( \frac{2}{3}yT_{ttt} + X_{tt} \right) u_y \\
& \left. - 2 \left( \frac{2}{3}yT_{tt} + X_t \right) u_{yt} \right] \frac{\partial \rho}{\partial u_{tt}} + T_t \rho = 0, \tag{61}
\end{aligned}$$

The solutions  $J_1$ ,  $J_2$  and  $\rho$  of (59)-(61) can be directly solved:

$$\rho = f_0(t)K_1(t_1, t_2, t_3, \dots, t_{12}), \tag{62}$$

$$\begin{aligned}
J_2 &= [f_1(t) + f_2(t)y]K_1(t_1, t_2, t_3, \dots, t_{12}) \\
&+ f_3(t)K_2(t_1, t_2, t_3, \dots, t_{12}), \tag{63}
\end{aligned}$$

$$\begin{aligned}
J_1 = & [f_4(t) + f_5(t)x + f_6(t)y + f_7(t)y^2]K_1(t_1, t_2, t_3, \dots, t_{12}) \\
& + [f_8(t) + f_9(t)y]K_2(t_1, t_2, t_3, \dots, t_{12}) \\
& + f_{10}(t)K_3(t_1, t_2, t_3, \dots, t_{12}),
\end{aligned} \tag{64}$$

where  $K_1, K_2$  and  $K_3$  are arbitrary functions of  $\{t_1, t_2, t_3, \dots, t_{12}\}$ , and  $f_i, i = 0, 1, \dots, 10$  are functions fixed by:

$$f_0 = T^{-1}, \tag{65}$$

$$f_1 = XT^{-2}, \quad f_2 = \frac{2}{3}T_tT^{-2}, \quad f_3 = T^{-\frac{4}{3}}, \tag{66}$$

$$\begin{aligned}
f_4 = YT^{-2}, \quad f_5 = \frac{1}{3}T_tT^{-2}, \quad f_6 = X_tT^{-2}, \quad f_7 = \frac{1}{3}T_{tt}T^{-2}, \\
f_8 = XT^{-\frac{7}{3}}, \quad f_9 = \frac{2}{3}T_tT^{-\frac{7}{3}}, \quad f_{10} = T^{-\frac{5}{3}},
\end{aligned} \tag{67}$$

with the invariants being

$$t_1 = T^{-\frac{2}{3}}y - X_1, \tag{68}$$

$$t_2 = T^{-\frac{1}{3}}x - \frac{1}{3}T^{-\frac{4}{3}}T_ty^2 - T^{-\frac{4}{3}}Xy - Y_1, \tag{69}$$

$$\begin{aligned}
t_3 = & -\frac{1}{3}T^{-\frac{2}{3}}T_tx^2 - \frac{10}{81}T^{-\frac{8}{3}}y^4T_t^3 - 2T^{-\frac{2}{3}}y^2Y_t - \frac{5}{3}T^{-\frac{8}{3}}yX^3 - T^{-\frac{2}{3}}yY_2 \\
& + T^{-\frac{5}{3}}xX^2 - 2T^{-\frac{2}{3}}xY + \frac{2}{9}T^{-\frac{5}{3}}y^4T_{tt}T_t + \frac{8}{9}T^{-\frac{5}{3}}y^3X_tT_t \\
& + \frac{4}{9}T^{-\frac{5}{3}}y^3XT_{tt} - 2T^{-\frac{2}{3}}yxX_t + \frac{4}{3}T^{-\frac{5}{3}}yxXT_t - \frac{2}{3}T^{-\frac{2}{3}}xy^2T_{tt} \\
& - \frac{1}{9}T^{-\frac{2}{3}}y^4T_{ttt} - \frac{2}{3}T^{-\frac{2}{3}}y^3X_{tt} - \frac{20}{27}T^{-\frac{8}{3}}y^3XT_t^2 + \frac{4}{9}T^{-\frac{5}{3}}xy^2T_t^2 \\
& + 2T^{-\frac{5}{3}}y^2XX_t - \frac{5}{3}T^{-\frac{8}{3}}y^2X^2T_t + \frac{4}{3}T^{-\frac{5}{3}}y^2T_tY + 4T^{-\frac{5}{3}}yXY \\
& + uT^{\frac{1}{3}} + \frac{1}{3}Y_3,
\end{aligned} \tag{70}$$

$$\begin{aligned}
t_4 = & u_xT^{\frac{2}{3}} - \frac{2}{3}T^{-\frac{1}{3}}T_tx + \frac{4}{9}T^{-\frac{4}{3}}y^2T_t^2 - \frac{2}{3}T^{-\frac{1}{3}}y^2T_{tt} - 2T^{-\frac{1}{3}}yX_t \\
& + \frac{4}{3}T^{-\frac{4}{3}}yXT_t + X^2T^{-\frac{4}{3}} - 2YT^{-\frac{1}{3}},
\end{aligned} \tag{71}$$

$$\begin{aligned}
t_5 = & \frac{2}{3}T_t u_x y + \frac{2}{3}T^{-1}xXT_t + \frac{4}{9}T^{-1}xyT_t^2 - \frac{4}{3}xyT_{tt} + \frac{4}{9}T^{-1}y^3T_tT_{tt} \\
& + \frac{4}{3}T^{-1}y^2X_tT_t + \frac{2}{3}T^{-1}y^2XT_{tt} - \frac{8}{9}T^{-2}y^2XT_t^2 + 2T^{-1}yXX_t \\
& - \frac{4}{3}T^{-2}yX^2T_t + \frac{4}{3}T^{-1}yYT_t + 2XT^{-1}Y + Xu_x - 2xX_t - \frac{4}{9}y^3T_{ttt} \\
& - 2y^2X_{tt} - 4yY_t - \frac{2}{3}X^3T^{-2} + u_yT - Y_2 - \frac{16}{81}T^{-2}y^3T_t^3,
\end{aligned} \tag{72}$$

$$\begin{aligned}
t_6 = & \frac{1}{9}T^{\frac{1}{3}}(9u_tT - 9Z_2 + 9u_yX + 9Yu_x - T_{ttt}y^4 - 3T_{tt}x^2 - 18Y_{tt}y^2 \\
& - 9Z_1y - 18Y_tx + 3T_tu - 6X_{ttt}y^3 + 6u_yT_t y + 3T_txu_x - 6xy^2T_{ttt} \\
& + 3T_{tt}y^2u_x + 9yu_xX_t - 18xX_{tt}y),
\end{aligned} \tag{73}$$

$$t_7 = u_{xx}T - \frac{2}{3}T_t, \tag{74}$$

$$\begin{aligned}
t_8 = & \frac{1}{9}T^{-\frac{2}{3}}(9u_{xy}T^2 + 9u_{xx}TX + 6u_{xy}T_tT + 4yT_t^2 - 12yTT_{tt} \\
& - 18TX_t + 6XT_t),
\end{aligned} \tag{75}$$

$$\begin{aligned}
t_9 = & \frac{1}{3}T^{\frac{2}{3}}(3u_{tx}T + 3u_{xy}X + 2u_{xy}yT_t + 3u_{xx}Y + 3u_{xy}X_t - 2y^2T_{ttt} \\
& + u_{xx}xT_t + u_{xx}y^2T_{tt} - 2xT_{tt} - 6Y_t + 2u_xT_t - 6X_{tt}y),
\end{aligned} \tag{76}$$

$$\begin{aligned}
t_{10} = & \frac{1}{27}T^{-\frac{4}{3}}(27u_{yy}T^3 + 54u_{xy}T^2X + 36u_{xy}yT^2T_t + 27u_{xx}X^2T \\
& + 36u_{xx}yTX_t + 12u_{xx}y^2T_t^2T - 108T^2Y_t - 18X^2T_t + 36TYT_t \\
& + 18T^2T_tu_x + 12xTT_t^2 - 36xT_{tt}T^2 - 8y^2T_t^3 + 12y^2TT_tT_{tt} \\
& - 36y^2T_{ttt}T^2 - 108yT^2X_{tt} + 36yTX_tT_t - 24yXT_t^2),
\end{aligned} \tag{77}$$

$$\begin{aligned}
t_{11} = & u_{ty}T^2 - Z_1T + u_{xy}X^2 - 2XY_t + \frac{4}{9}u_{xy}T_t^2 + \frac{2}{3}u_xXT_t \\
& - \frac{4}{3}yY_tT_t + u_xTX_t + u_{yy}TX + u_{tx}TX + \frac{4}{9}u_{xy}y^2T_t^2 \\
& - \frac{2}{3}xXT_{tt} - \frac{2}{3}y^2XT_{ttt} - 2yXX_{tt} - \frac{4}{9}y^3T_{ttt}T_t - \frac{4}{3}y^2X_{tt}T_t \\
& - 2xTX_{tt} - 2y^2TX_{ttt} - \frac{4}{9}y^3TT_{ttt} - 4yTY_{tt} + u_{xy}TY \\
& + u_{xx}XY + \frac{1}{3}TT_tu_{xy}x + u_{xy}yTX_t + \frac{1}{3}u_{xy}y^2TT_{tt} - \frac{4}{3}xyTT_{ttt} \\
& + \frac{2}{3}u_{xy}yTT_{tt} + u_{xx}yXX_t + \frac{2}{3}u_{xx}yT_tY + \frac{1}{3}u_{xx}y^2XT_{tt} \\
& + \frac{2}{3}u_{xx}y^2T_tX_t + \frac{4}{3}u_{xy}yXT_t - \frac{4}{9}xyT_{tt}T_t + \frac{2}{9}u_{xx}y^3T_{tt}T_t \\
& + \frac{1}{3}u_{xx}xXT_t + \frac{2}{9}u_{xx}xyT_t^2 + \frac{2}{3}TT_tu_{yy}y + \frac{2}{3}u_{tx}TT_t y + TT_tu_y,
\end{aligned} \tag{78}$$



$$\begin{aligned}
t_{12} = & \frac{1}{9}T^{\frac{1}{3}}(-9Z_{2,t}T - 9Z_1X + uT_t^2 + 15u_xyX_tT_t + 9u_{tt}T^2 \\
& + 9u_{yy}X^2 + 9u_{xx}Y^2 + 15u_tT_tT - 18Y_tY + 18u_{ty}TX \\
& + 18u_{tx}TY - 6y^2T_{tt}Y_t - 6y^2T_{ttt}Y + 3uTT_{tt} - 18y^2TY_{ttt} \\
& - 6y^3TX_{ttt} - 18xTY_{tt} - Ty^4T_{tttt} - 3Tx^2T_{ttt} - 14y^3X_{ttt}T_t \\
& - 4y^3XT_{ttt} - 18xXX_{tt} - 6xT_{tt}Y - 18y^2X_{tt}X_t - 30y^2Y_{tt}T_t \\
& + 9u_{xx}y^2X_t^2 - 6y^3T_{tt}X_{tt} - 6y^3T_{ttt}X_t - 2y^4T_{tt}T_{ttt} - 3y^4T_tT_{ttt} \\
& - 3x^2T_{tt}T_t + u_{xx}y^4T_{tt} + 18u_{xy}XY + u_{xx}x^2T_t^2 + 4u_{yy}y^2T_t^2 \\
& - 18yYX_{tt} - 36yXY_{tt} - 2xy^2T_{tt}^2 - 18y^2XX_{ttt} - 3Z_2T_t \\
& + 2u_{xx}xy^2T_{tt}T_t + 6u_{xx}xyT_tX_t - 9yTZ_{1,t} + 9u_yTX_t + 9u_xTY_t \\
& - 12xY_tT_t + 12u_yXT_t + 9u_xXX_t + 9u_xT_tY - 18yY_tX_t - 9yZ_1T_t \\
& + 3u_xXT_t^2 + 8u_yyT_t^2 + 9u_xyTX_{tt} + 6u_{tx}y^2TT_{tt} + 6u_{tx}xT_tT \\
& + 12u_{ty}yT_tT - 18xyTX_{ttt} + 3Tu_xy^2T_{ttt} + 18u_{tx}yTX_t \\
& + 3u_xXT_{tt} + 6u_yyTT_{tt} - 6Txy^2T_{ttt} + 12u_{xy}y^2T_tX_t \\
& + 6u_{xy}y^2XT_{tt} + 6u_xyXT_{tt} + 4u_{xy}xyT_t^2 + 4u_{xy}y^3T_{tt}T_t \\
& + 6u_{xy}xXT_t + 12u_{yy}yXT_t + 6u_{xx}xYT_t + 6u_{xx}y^3T_{tt}X_t \\
& + 18u_{xx}yYX_t - 24xyX_{tt}T_t - 6xyX_tT_{tt} - 12xyXT_{ttt} \\
& + 7u_xy^2T_{tt}T_t - 12xy^2T_{ttt}T_t + 6u_{xx}y^2YT_{tt} \\
& + 18u_{xy}yXX_t + 12u_{xy}yYT_t), \tag{79}
\end{aligned}$$

where

$$X_{1t} = XT^{-\frac{5}{3}}, \tag{80}$$

$$Y_{1t} = -T^{-\frac{7}{3}}X^2 + T^{-\frac{4}{3}}Y, \tag{81}$$

$$Y_{2t} = Z_1, \tag{82}$$

$$Y_{3t} = -T^{-\frac{11}{3}}(15YX^2T - 6Y^2T^2 + 3Z_2T^3 - 5X^4 - 3XY_2T^2). \tag{83}$$

To determine the functions of  $K_1$ ,  $K_2$ , and  $K_3$ , we substitute (62), (63), and (64) into

(58) which yields a complicated equation

$$\begin{aligned}
& J_{1,x} + J_{1,u}u_x + J_{1,u_x}u_{xx} + J_{1,u_y}u_{xy} + J_{1,u_t}u_{xt} + J_{1,u_{xx}}u_{xxx} \\
& + J_{1,u_{xy}}u_{xxy} + J_{1,u_{xt}}u_{xxt} + J_{1,u_{yy}}u_{yy} + J_{1,u_{yt}}u_{xyt} + J_{1,u_{tt}}u_{xtt} \\
& + J_{2,y} + J_{2,u}u_y + J_{2,u_x}u_{xy} + J_{2,u_y}u_{yy} + J_{2,u_t}u_{yt} + J_{2,u_{xx}}u_{xxy} \\
& + J_{2,u_{xy}}u_{xyy} + J_{2,u_{xt}}u_{xyt} + J_{2,u_{yy}}u_{yyy} + J_{2,u_{yt}}u_{yyt} + J_{2,u_{tt}}u_{ytt} \\
& + \rho_t + \rho_u u_t + \rho_{u_x}u_{xt} + \rho_{u_y}u_{yt} + \rho_{u_t}u_{tt} + \rho_{u_{xx}}u_{xxt} + \rho_{u_{xy}}u_{xyt} \\
& + \rho_{u_{xt}}u_{xtt} + \rho_{u_{yy}}u_{yyt} + \rho_{u_{yt}}u_{ytt} + \rho_{u_{tt}}u_{ttt} = 0.
\end{aligned} \tag{84}$$

To solve the complicated equation (84), we begin from the highest derivatives of  $u$  for  $K_1$ ,  $K_2$  and  $K_3$  being  $\{u_{xxx}, u_{xxy}, \dots, u_{ttt}\}$  independent. Letting the coefficients of  $\{u_{xxx}, u_{xxy}, \dots, u_{ttt}\}$  be zero, we can get a more simplified equation. For example, the term of  $u_{ttt}$  in (84) is

$$3T^3 K_{1,t_{12}} u_{ttt}, \tag{85}$$

where  $K_{i,t_j^n} = \frac{\partial^j K_i}{\partial t_j^n}$ . There is no nontrivial solution for  $K_{1,t_{12}} \neq 0$ , thus the only possible case is

$$K_{1,t_{12}} = 0, \quad i.e., \quad K_1 \equiv K_1(t_1, t_2, \dots, t_{11}). \tag{86}$$

Under (86), the coefficient of  $u_{ytt}$  is

$$3T^{\frac{8}{3}}(K_{2,t_{12}} + K_{1,t_{11}}), \tag{87}$$

which leads to the only possible solution

$$K_2(t_1, t_2, t_3, \dots, t_{12}) = -t_{12}K_{1,t_{11}} + K_{21}(t_1, t_2, t_3, \dots, t_{11}), \tag{88}$$

with  $K_{21}(t_1, t_2, t_3, \dots, t_{11})$  being an un-determined function of the indicated variables.

Like the procedure to eliminate  $u_{ttt}$  and  $u_{ytt}$ , vanishing the terms of  $u_{xxx}, u_{xxy}, \dots, u_{yyy}$  results in

$$\begin{aligned}
K_1 = & (-F_2 t_8 + F_3 t_7 + F_{10})t_{11} + (F_1 t_7 + F_{11} + F_2 t_9)t_{10} \\
& + (F_4 - t_8 F_3)t_9 - F_1 t_8^2 + (F_5 + F_6)t_8 + F_8 t_7 + F_{14},
\end{aligned} \tag{89}$$

$$\begin{aligned}
K_2 = & (F_2 t_8 - F_3 t_7 - F_{10})t_{12} + (-F_1 t_7 - F_2 t_9 - F_{11})t_{11} \\
& + t_9^2 F_3 + (F_5 + F_1 t_8)t_9 + F_7 t_7 + F_9 t_8 + F_{13},
\end{aligned} \tag{90}$$

$$\begin{aligned}
K_3 = & (-F_2 t_{10} + t_8 F_3 - F_4) t_{12} + F_2 t_{11}^2 + (F_1 t_8 - F_3 t_9 - 2F_5 \\
& - F_6) t_{11} + (-F_1 t_9 - F_9) t_{10} - F_7 t_8 - F_8 t_9 + F_{12},
\end{aligned} \tag{91}$$

with 14 equations satisfied:

$$\begin{aligned}
& F_{1,t_3} t_6 + F_{7,t_5} - F_{9,t_4} = 0, \\
& -F_{2,t_4} + F_{1,t_6} - F_{3,t_5} = 0, \\
& F_{8,t_3} t_6 + F_{12,t_4} + F_{7,t_1} + F_{7,t_3} t_5 = 0, \\
& -F_{4,t_4} + F_{8,t_6} - F_{3,t_3} t_5 - F_{3,t_1} = 0, \\
& -F_{9,t_2} + F_{13,t_5} - F_{9,t_3} t_4 + F_{11,t_3} t_6 = 0, \\
& -F_{11,t_6} + F_{2,t_2} + F_{10,t_5} + F_{2,t_3} t_4 = 0, \\
& -F_{10,t_1} - F_{4,t_3} t_4 + F_{14,t_6} - F_{10,t_3} t_5 - F_{4,t_2} = 0, \\
& F_{13,t_1} + F_{12,t_3} t_4 + F_{13,t_3} t_5 + F_{14,t_3} t_6 + F_{12,t_2} = 0, \\
& -F_{5,t_5} - F_{11,t_4} + F_{1,t_3} t_4 + F_{1,t_2} - F_{2,t_3} t_6 + F_{9,t_6} = 0, \\
& F_{14,t_4} + F_{5,t_3} t_5 + F_{5,t_1} + F_{12,t_6} - F_{8,t_3} t_4 - F_{8,t_2} + F_{4,t_3} t_6 = 0, \\
& -F_{8,t_5} - F_{7,t_6} + 2F_{5,t_4} + F_{1,t_3} t_5 - F_{3,t_3} t_6 + F_{6,t_4} + F_{1,t_1} = 0, \\
& -F_{4,t_5} + F_{2,t_3} t_5 - F_{10,t_4} + F_{2,t_1} + F_{3,t_3} t_4 + F_{6,t_6} + F_{5,t_6} \\
& + F_{3,t_2} = 0, \\
& (F_{5,t_3} + F_{6,t_3}) t_6 + F_{9,t_1} - F_{7,t_3} t_4 + F_{12,t_5} + F_{9,t_3} t_5 + F_{13,t_4} \\
& - F_{7,t_2} = 0, \\
& F_{10,t_3} t_6 - (F_{6,t_3} + 2F_{5,t_3}) t_4 + F_{13,t_6} + F_{14,t_5} - F_{11,t_1} - 2F_{5,t_2} \\
& - F_{11,t_3} t_5 - F_{6,t_2} = 0,
\end{aligned} \tag{92}$$

where  $F_i$ ,  $i = 1, 2, \dots, 14$  are functions of  $\{t_1, t_2, t_3, t_4, t_5, t_6\}$ . Fortunately, the equations are linear system which is straightforward to solve.

For simplicity, we just list the final solutions:

$$\begin{aligned}
K_1 = & (\alpha_{2,t_6}t_7 + \alpha_{5,t_6} - \alpha_{1,t_5t_6}t_8)t_{11} + (\alpha_{1,t_5t_6}t_9 + (\alpha_{1,t_4t_5} \\
& + \alpha_{2,t_5} + \alpha_{6,t_5t_5})t_7 + \alpha_{5,t_5} + \alpha_{7,t_5t_5} + \alpha_{1,t_2t_5} + t_4\alpha_{1,t_3t_5})t_{10} \\
& + (-\alpha_{2,t_6}t_8 + \alpha_{3,t_6})t_9 + (-\alpha_{1,t_4t_5} - \alpha_{2,t_5} - \alpha_{6,t_5t_5})t_8^2 \\
& + (\alpha_{5,t_4} + \alpha_{8,t_5} - \alpha_{1,t_1t_5} - t_4\alpha_{2,t_3} - t_5\alpha_{1,t_3t_5} + \alpha_{3,t_5} - \alpha_{2,t_2})t_8 \\
& + ((\alpha_{6,t_3t_5} + \alpha_{2,t_3} - \alpha_{10,t_4t_4})t_5 + \alpha_{6,t_2t_4} + \alpha_{11,t_4} + t_4\alpha_{6,t_3t_4} \\
& + \alpha_{3,t_4} - \alpha_{7,t_4t_4} + \alpha_{6,t_3} + \alpha_{2,t_1} + \alpha_{6,t_1t_5} + \alpha_{8,t_4})t_7 + (\alpha_{7,t_3t_5} \\
& - \alpha_{10,t_2t_4} + \alpha_{15,t_3} - t_4\alpha_{10,t_3t_4} + \alpha_{10,t_3} + \alpha_{5,t_3})t_5 + t_4^2\alpha_{6,t_3t_3} \\
& + (\alpha_{13,t_3} + \alpha_{11,t_3} - \alpha_{7,t_3t_4} + \alpha_{8,t_3} + \alpha_{3,t_3} + 2\alpha_{6,t_2t_3})t_4 \\
& + \alpha_{17} + \alpha_{5,t_1} + \alpha_{7,t_1t_5} + \alpha_{3,t_2} + \alpha_{8,t_2} + \alpha_{10,t_1} + \alpha_{15,t_1} \\
& + \alpha_{6,t_2t_2} - \alpha_{7,t_2t_4} + \alpha_{11,t_2} + \alpha_{13,t_2}, \tag{93}
\end{aligned}$$

$$\begin{aligned}
K_2 = & (\alpha_{1,t_5t_6}t_8 - \alpha_{2,t_6}t_7 - \alpha_{5,t_6})t_{12} + (-\alpha_{1,t_5t_6}t_9 \\
& + (-\alpha_{1,t_4t_5} - \alpha_{2,t_5} - \alpha_{6,t_5t_5})t_7 - \alpha_{7,t_5t_5} - \alpha_{1,t_2t_5} \\
& - t_4\alpha_{1,t_3t_5} - \alpha_{5,t_5})t_{11} + t_9^2\alpha_{2,t_6} + (t_8(\alpha_{1,t_4t_5} \\
& + \alpha_{2,t_5} + \alpha_{6,t_5t_5}) - t_6\alpha_{1,t_3t_6} + (\alpha_{6,t_3t_5} + \alpha_{2,t_3})t_4 \\
& - \alpha_{7,t_4t_5} + \alpha_{9,t_6} - \alpha_{1,t_3} - \alpha_{5,t_4} + \alpha_{6,t_2t_5} + \alpha_{4,t_6} \\
& + \alpha_{2,t_2})t_9 + \alpha_{4,t_5}t_8 + ((\alpha_{10,t_4t_4} - \alpha_{2,t_3} - \alpha_{6,t_3t_5} \\
& - \alpha_{1,t_3t_4})t_6 + \alpha_{4,t_4} + \alpha_{9,t_4} + \alpha_{12,t_4})t_7 \\
& + ((-\alpha_{1,t_3t_3} + \alpha_{10,t_3t_4})t_4 - \alpha_{10,t_3} - \alpha_{5,t_3} - \alpha_{7,t_3t_5} \\
& - \alpha_{1,t_2t_3} + \alpha_{10,t_2t_4} - \alpha_{15,t_3})t_6 + \alpha_{9,t_2} + \alpha_{4,t_2} + \alpha_{12,t_2} \\
& + (\alpha_{12,t_3} + \alpha_{14,t_3} + \alpha_{9,t_3} + \alpha_{4,t_3})t_4 + \alpha_{14,t_2} \\
& - \alpha_{16,t_2} + \alpha_{18}, \tag{94}
\end{aligned}$$

$$\begin{aligned}
K_3 = & (-\alpha_{1,t_5t_6}t_{10} + \alpha_{2,t_6}t_8 - \alpha_{3,t_6})t_{12} + t_{11}^2\alpha_{1,t_5t_6} \\
& + (t_8(\alpha_{1,t_4t_5} + \alpha_{2,t_5} + \alpha_{6,t_5t_5}) - \alpha_{2,t_6}t_9 - t_4\alpha_{6,t_3t_5} \\
& + \alpha_{1,t_3} + \alpha_{7,t_4t_5} + t_6\alpha_{1,t_3t_6} - \alpha_{4,t_6} - \alpha_{6,t_2t_5} - \alpha_{9,t_6} \\
& - \alpha_{3,t_5} + \alpha_{1,t_1t_5} + t_5\alpha_{1,t_3t_5} - \alpha_{8,t_5})t_{11} + ((-\alpha_{1,t_4t_5} \\
& - \alpha_{2,t_5} - \alpha_{6,t_5t_5})t_9 - \alpha_{4,t_5})t_{10} + ((-\alpha_{2,t_3} - \alpha_{6,t_3t_5} \\
& + \alpha_{10,t_4t_4})t_5 - \alpha_{2,t_1} - \alpha_{3,t_4} - \alpha_{6,t_2t_4} - \alpha_{6,t_3} - \alpha_{8,t_4} \\
& + \alpha_{7,t_4t_4} - \alpha_{6,t_1t_5} - t_4\alpha_{6,t_3t_4} - \alpha_{11,t_4})t_9 \\
& + ((-\alpha_{10,t_4t_4} + \alpha_{2,t_3} + \alpha_{6,t_3t_5} + \alpha_{1,t_3t_4})t_6 - \alpha_{4,t_4} \\
& - \alpha_{9,t_4} - \alpha_{12,t_4})t_8 + (t_5\alpha_{1,t_3t_3} - t_4\alpha_{6,t_3t_3} - \alpha_{6,t_2t_3} \\
& - \alpha_{8,t_3} - \alpha_{3,t_3} - \alpha_{10,t_1t_4} - \alpha_{11,t_3} - \alpha_{13,t_3} + \alpha_{1,t_1t_3} \\
& + \alpha_{7,t_3t_4})t_6 + (-\alpha_{9,t_3} - \alpha_{4,t_3} - \alpha_{12,t_3} - \alpha_{14,t_3})t_5 \\
& - \alpha_{4,t_1} - \alpha_{12,t_1} - \alpha_{14,t_1} + \alpha_{16,t_1} - \alpha_{9,t_1}, \tag{95}
\end{aligned}$$

with  $\alpha_i (i = 1, 2, \dots, 5)$  being arbitrary functions of  $\{t_1, t_2, t_3, t_4, t_5, t_6\}$ ,  $\alpha_i (i = 6, 7, 8)$  being arbitrary functions of  $\{t_1, t_2, t_3, t_4, t_5\}$ ,  $\alpha_9$  being arbitrary function of  $\{t_1, t_2, t_3, t_4, t_6\}$ ,  $\alpha_i (i = 10, 11, 12)$  being arbitrary functions of  $\{t_1, t_2, t_3, t_4\}$ ,  $\alpha_{13}, \alpha_{14}, \alpha_{15}$  being arbitrary functions of  $\{t_1, t_2, t_3\}$ ,  $\alpha_{16}, \alpha_{17}$  being arbitrary functions of  $\{t_1, t_2\}$  and  $\alpha_{18}$  is function of  $t_2$ .

Substituting (93)-(95) into (62)-(64), we can obtain the conservation laws of the Lin-Tsien equation associated with the Lie-Bäcklund generator  $X_0$ . We have verified that the conserved vector  $(J_1, J_2, \rho)$  really satisfied Eq. (58).

#### IV. CONCLUSION AND DISCUSSION

In this paper, by applying the modified CK's direct method, we set up Theorem 1, which shows the relationship between new exact solutions and old ones of the (2+1)-dimensional Lin-Tsien equation. Using the transformation relations we then get the corresponding KMV symmetry algebra and the Lie point symmetry which coincide with the result generated from the standard Lie approach.

We generate the conservation laws of the Lin-Tsien equation related to the infinite dimensional KMV symmetry group by use of the Lie-Bäcklund generator up to the second

order group invariants. The existence of arbitrary functions of the group invariants proves the Lin-Tsien equation has infinitely many conservation laws which connect with the general Lie point symmetry (37). Though the symmetries and conservation laws are obtained from the Lin-Tsien equation, the conservation laws we derived is only depended on the symmetry which may be possessed by many equations.

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